The Art of the Wunderlich Cube and the Development of Spatial Abilities

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Abstract
This paper advocates for a future where the teaching of math and art are harmoniously intertwined as they were in the days of da Vinci. In this future, *code* provides the "brush" that enables the expression of artistic ideas and mathematical structures in digital and digitally fabricated mediums. This educational idea is motivated by (1) literature supporting the position that visual thinking and spatial reasoning significantly impact STEAM disciplines, and (2) Piaget's theory of cognitive development in which children, in the concrete operational stage, solve problems relating to physical objects (i.e., they learn-by-making). We then describe a project involving the creation of a 3D artifact we call a *Wunderlich cube* – a mathematical artifact that embodies numerous spatial reasoning puzzles. An understanding of the properties of the Wunderlich cube is developed through manual construction using LEGO®, mathematical analysis, computational thinking, coding, and 3D printing.

Author Keywords
coding; functional programming; spatial reasoning; visual thinking; space-filling curve; LEGO®; 3D printing;

ACM Classification Keywords
K.3.2 [Computer and Information Science Education]: Computer science education;

Figure 1: Leonardo da Vinci: "*Let no man who is not a Mathematician read the elements of my work.*"

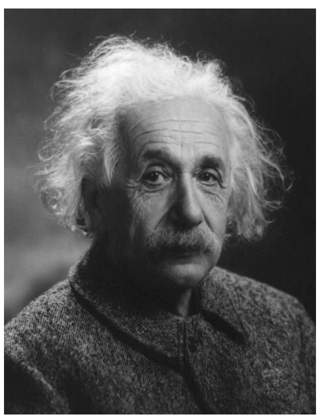

Figure 2: Albert Einstein: *"[Words] do not seem to play any role in my mechanism of thought.*"

Motivation

There is growing agreement that coding will be fundamental to technologically advanced societies of the 21st century. In effect, coding is the "pencil" of the 21st century. Countries such as England, Australia, and Finland have all mandated that coding be taught across the entire K12 spectrum. However, incorporating coding into school curricula that are already overfull represents a serious challenge. In Australia, time to teach coding was made by eliminating history and geology from the curriculum [3]. In the US, a more harmonious approach is advocated by the Bootstrap group in which the teaching of coding and math are tightly coupled [4]. In effect, coding is seen as a medium by which certain mathematical concepts can be taught. In such an approach, pressure to make room in a curriculum is greatly reduced. This mindset can be used in a larger context to consider potential synergies between coding and a variety of other disciplines. Central to such an approach is the identification of thought processes that span a variety of disciplines such as coding, art, math, and science.

Spatial Abilities

It is often said that designing and developing code involves thought processes related to logic and mathematical reasoning. What is not so often said is that a variety of thought processes that underly coding also underly art, and that ultimately coding, math, and art are more intertwined than people might realize. In his fascinating book on Φ, Mario Livio journeys through a history of math – a history filled with artists. In the time of Leonardo da Vinci, fluency in math was essentially a prerequisite to the study of art as evidenced by the following quote.

> "*Let no man who is not a Mathematician read the elements of my work.*"
>
> Leonardo da Vinci,
> Prolegomena and General
> Introduction to the Book
> on Painting

More recently, a study has found that training in visual arts is a significant predictor of abilities relating to the manipulation of artifacts in two dimensional and three dimensional space [9] – a finding that would probably have been self-evident to the artists of da Vinci's time. In similar spirit, Einstein described his thought processes as visually based, saying that words "do not seem to play any role in my mechanism of thought". To Einstein, the difference between science and art lay in the medium that was used to express ideas.

> "*If what is seen and experienced is portrayed in the language of logic, then it is science. If it is communicated through forms whose constructions are not accessible to the conscious mind but are recognized intuitively, then it is art*" [2].

In 2009, Wai et al, published the results of an 11+ year longitudinal study involving a large student population in which they found that spatial ability played a critical role in the development of STEM abilities [8]. Analysis performed by Newcombe supports the position that spatial abilities can be developed through practice, especially through the visual arts [6]. Newcombe goes on to argue that spatial reasoning is important to a wide range of disciplines including chemistry, social studies, and physics. She endorses a strategy recommended in the *Learning to Think Spatially* report which calls for existing curriculum to be "spatialized". Another study involving 8 year old students, conducted by

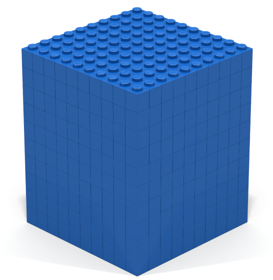

Figure 3: A $10 \times 10 \times 10$ cube created by a Bricklayer program.

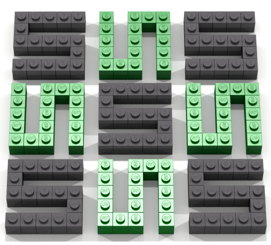

Figure 4: A checkerboard-like tiling of a $3^1 \times 3^1$ cell grid.

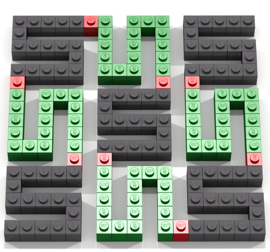

Figure 5: A type 1 Wunderlich curve spanning a $3^1 \times 3^1$ cell grid.

Gunderson et al, concluded that there exists a "strong relationship between spatial skills, number line knowledge, and math achievement" [5].

Well-known child psychologist Jean Piaget identified four stages of cognitive development [1]. In this theory the third stage, occurring between ages 7 – 11, is called the concrete operational stage. A key characteristic of this stage is that cognitive development is limited to analyzing and developing an understanding of artifacts having physical manifestations. A somewhat oversimplified characterization of this stage is that children "learn-by-making". We further believe that interaction with graspable objects has significant benefits for people of all ages, and that digital fabrication can provide a wonderful connection between the digital world and the physical world.

The "Brush"

Bricklayer [10] is an online educational ecosystem designed in accordance with a "low-threshold infinite-ceiling" philosophy. Its purpose is to teach coding to people of all ages and coding backgrounds. A significant portion of the Bricklayer ecosystem has been developed specifically to help novices, especially primary school children, learn how to code. When executed, Bricklayer programs can produce LEGO® artifacts, Minecraft artifacts, as well as artifacts suitable for 3D printing. Bricklayer resides in a domain in which there is a strong connection between math, art, and computer science. The Bricklayer ecosystem is freely-available and can be found at:

http://bricklayer.org

Bricklayer programs are written in the functional programming language SML. Graphical capabilities are provided by the Bricklayer library. The output of Bricklayer programs are files which are input to third-party tools which include: LEGO® Digital Designer (LDD), LDraw, Minecraft, and STL viewers such as 3D Builder. Figure 6 shows the code for a Bricklayer program that creates a $10 \times 10 \times 10$ LEGO® cube.

```
open Level_4;

build(10,10,10);

put (10,10,10) BLUE (0,0,0);

show "A cube.";
```

Figure 6: A Bricklayer program that creates a cube.

Wunderlich Curves

Walter Wunderlich is attributed with the discovery of three space-filling curves called Wunderlich curves [7]. In contrast to the Hilbert curve, which involves a square of size $2^n \times 2^n$, Wunderlich curves involve squares of size $3^n \times 3^n$. A Wunderlich curve is constructed using rotations and reflections of an initial seed shape to create patterns which can be connected to form a space-filling curve. Wunderlich curves are suitable for iterative as well as recursive construction algorithms. In this article we take a look at the first Wunderlich curve, which we will refer to as a *type 1 Wunderlich curve*.

A type 1 Wunderlich curve is constructed using a seed shape similar to the letter S. Using this seed shape and its $90°$ rotation, a $3^1 \times 3^1$ cell grid can be tiled in a checkerboard-like fashion as shown in Figure 4. The shapes in the tiling can then be connected, as shown in Figure 5, to create a type 1 Wunderlich curve. A larger Wunderlich curve can then be created by using the $3^1 \times 3^1$ pattern as the seed pattern to "tile" a $3^2 \times 3^2$ cell grid.

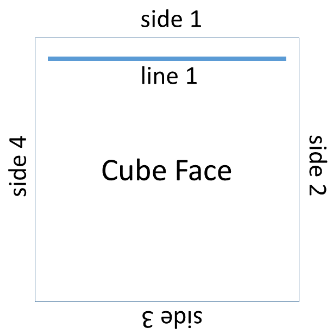

Figure 7: An example of a line which is parallel and perpendicular to the sides of a cube face.

Some Cube Concepts and Terminology

An $n \times n \times n$ cube is a 3D artifact whose surface consists of 6 faces. A face of an $n \times n \times n$ cube is an $n \times n$ square. The perimeter of a square consists of 4 line segments called sides. Two faces, f_1 and f_2, of a cube are adjacent if they share a side. We will also refer to the region that is shared as a seam.

Definition 1 *We are given a cube face f_1 having sides $s_1, s_2, s_3,$ and s_4 on which a line segment $line_1$ is drawn. We say $line_1$ is parallel to a side s_i, if $line_1$ is parallel to s_i. We say $line_1$ is perpendicular to s_i, if $line_1$ is perpendicular to s_i.*

In Figure 7, the line segment $line_1$ is parallel to $side_1$ and $side_3$, and perpendicular to $side_2$ and $side_4$.

We define an $n \times m$ cell grid, $\mathcal{G}$, as a rectangular arrangement of cells. Within a cell grid, cell positions are denoted by tuples of the form (i, j) as shown in Figure 8.

(0,m)				(n,m)
$\vdots$				
(0,1)	(1,1)	(2,1)		
(0,0)	(1,0)	(2,0)	$\cdots$	(n,0)

Figure 8: Cell coordinates.

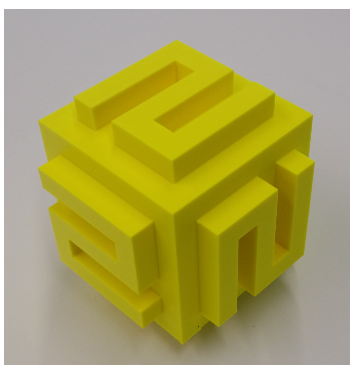

Figure 9: A 3D printed type 1 Wunderlich cube.

A *cube-path* over $\mathcal{G}$ is a sequence of cells created when a cube, whose faces have the same dimensions as the cells in $\mathcal{G}$, is "rolled" over one or more cells in the grid. If a cell is touched by a cube face, it is added to the cube-path. Rolling a cube is conceptually similar to rolling a cylinder or a sphere. Specifically, a cube is rolled from one cell to another by rotating the cube $90°$. Cubes can only be rolled vertically or horizontally (they cannot be rolled diagonally).

Example. The cube-path $(0,0) \rightarrow (1,0) \rightarrow (2,0)$ describes a path which begins at the cell whose coordinate is $(0,0)$, passes through the cell whose coordinate is $(1,0)$ and ends at the cell whose coordinate is $(2,0)$.

Definition 2 *A cube-path is $complete$ with respect to a given grid $\mathcal{G}$ if it contains all the cells in $\mathcal{G}$.*

Definition 3 *An $Eulerian\ cube\text{-}path$ is a cube-path which is complete and minimal (i.e., each cell only occurs once in the path).*

Definition 4 *A $stamp\ cube$ is a cube whose faces contain raised stamp shapes which can be used to create reflected (i.e., mirrored) impressions on 2D surfaces such as cell grids.*

The Definition of a Type 1 Wunderlich Cube

A type 1 Wunderlich cube is a stamp cube whose faces contain raised stamp shapes corresponding to reflections of the seed shape (S) and reflections of its $90°$ rotation. If the cube is constructed properly, it is possible to create a complete cube-path covering the $3^1 \times 3^1$ cell grid leaving on the grid the indentation of a pattern similar to that shown in Figure 4.

Definition 5 *Let f_1 and f_2 denote two adjacent faces of a stamp cube which share the seam s. Let p_1 denote the end point of the line segment of the stamp shape on f_1 that is closest to s. Similarly, let p_2 denote the end point of the line*

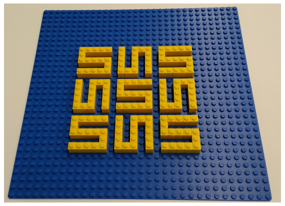

Figure 10: A LEGO prototype of the type 1 Wunderlich pattern.

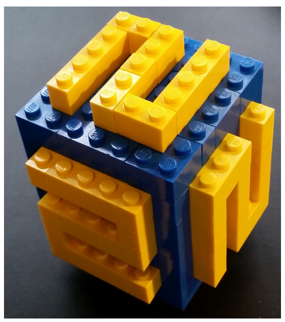

Figure 11: A LEGO prototype of the type 1 Wunderlich pattern.

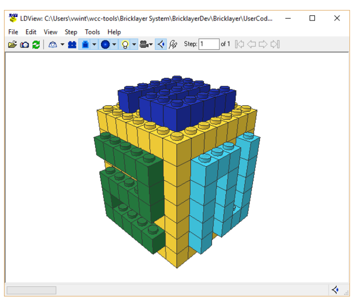

Figure 12: An LDraw visualization of a Bricklayer artifact.

segment of the stamp shape on f_2 that is closest to s. The faces f_1 and f_2 satisfy the type 1 Wunderlich property *if the following conditions hold:*

- *The line segment containing p_1 or the line segment containing p_2 is parallel to s.*
- *The line segment containing p_1 or the line segment containing p_2 is perpendicular to s.*

Theorem 1 *A cube is a* type 1 Wunderlich cube *if the following properties hold.*

1. *Each face contains a raised stamp shape corresponding to the reflection of the seed shape (S) or the reflection of the $90°$ rotation of the seed shape.*
2. *All pairs of adjacent faces in the cube possess the* type 1 Wunderlich property*.*

Proof: Adjacent cells in a cube-path correspond to adjacent cube faces. In a type 1 Wunderlich cube, all pairs of adjacent faces have the type 1 Wunderlich property. □

Corollary 1 *Any complete cube-path over a $3^n \times 3^n$ cell grid made by a type 1 Wunderlich cube will produce a type 1 Wunderlich pattern.*

A question that is not answered by the above theorem is whether it is possible to construct a type 1 Wunderlich cube in physical space. For the curious reader, the answer is "yes". Figures 9 and 11 respectively show a 3D printed version of a type 1 Wunderlich cube created using Bricklayer software and a LEGO® version.

For the rest of this paper we will refer to a type 1 Wunderlich cube as simply a Wunderlich cube.

Construction Projects

A variety of projects relating to the (type 1) Wunderlich cube are possible. Which projects are undertaken depends on the target audience as well as available time and resources.

Project: "Unplugged" – using LEGO and paper and pencil

1. Build a $3^1 \times 3^1$ Wunderlich pattern similar to the one shown in Figure 10.
2. Build a $3^1 \times 3^1$ Wunderlich curve similar to the one shown in Figure 5.
3. Figure 13 shows the flattened form of a cube – a typical starting point for a variety of spatial reasoning exercises. What stamp patterns (i.e., the reflection of the seed shape (S) or the reflection of the $90°$ rotation of the seed shape) need to be placed in the squares of this flattened cube so that the corresponding 3D cube is a Wunderlich cube? In the first experiment, place the reflection of the seed shape (S) at the center square of the cross shape. In the second experiment, place the reflection of the $90°$ rotation of the seed shape at the center square of the cross shape. Question: Does a Wunderlich cube have the same number of reflections of seed shapes and reflections of $90°$ rotations of the seed shape?

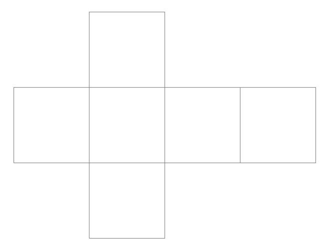

Figure 13: A flattened cube.

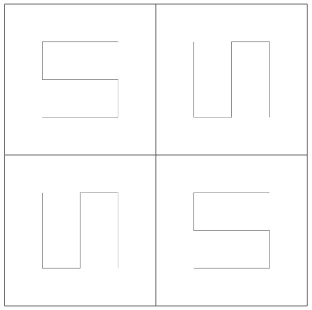

Figure 14: An Eulerian cube-path over an $n \times n$ cell grid, constructed using a Wunderlich cube, which can be connected to form a curve.

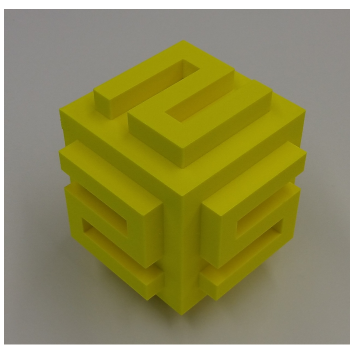

Figure 15: A mystery cube.

4. Build a Wunderlich cube similar to the one shown in Figure 11.

5. Using a suitable sized cookie tray, create a cell grid using Oobleck, Crayola® Model Magic, or Moon sand (see Figure 16). Create a complete cube-path in the grid to validate that the LEGO® cube constructed is indeed a Wunderlich cube.

It is worth noting that the paper-an-pencil exercise described here is more spontaneous (and therefore more error prone) than the LEGO® exercises, which are more deliberate.

Project: "Plugged-in" – using Bricklayer

1. Write a program that creates a $3^1 \times 3^1$ Wunderlich pattern similar to the one shown in Figure 4.

2. Write a program that creates a $3^1 \times 3^1$ Wunderlich curve similar to the one shown in Figure 5.

3. Write programs that create $3^2 \times 3^2$ and $3^3 \times 3^3$ Wunderlich curves.

4. Write a program that creates a virtual Wunderlich cube. Use LDraw to visually validate the created artifact. In this case, viewing the artifact using LDraw is better than using LEGO® Digital Designer (LDD). The reason for this is that in LDraw it is possible to have continual rotations around a given axis (e.g., x-axis, y-axis, and z-axis). This allows for a more accurate validation of the artifact than would be possible using LDD which only allows a half rotation around the x-axis.

5. Use the STL output file produced by Bricklayer to produce a 3D printed version of the Wunderlich cube created by your Bricklayer program. **Important:** Be certain your artifact is a Wunderlich cube since 3D printing can be quite expensive and time consuming!

Variations - What to do with failed attempts?

Stamp cubes can be created in which each face contains a raised stamp shape corresponding to the reflection of the seed shape (S) or the reflection of the a $90°$ rotation of the seed shape, which are nevertheless not Wunderlich cubes. If care isn't taken such cubes can easily be constructed. Furthermore, if such a cube is 3D printed considerable resources have been used. We affectionately call such cubes *mystery cubes*. Rather than discarding a mystery cube it is interesting to explore its properties. Figure 15 shows a mystery cube.

For a given mystery cube, answer the following questions and do the following activities.

1. Does there exist a complete cube-path that creates a pattern which can be connected to form a space-filling curve?

2. Do all complete cube-paths generated by the mystery cube produce the same pattern?

3. Create a LEGO® representation of the pattern created by this mystery cube.

4. Write a Bricklayer program that creates such a curve for cell grids of size $3^1 \times 3^1$, $3^2 \times 3^2$, and $3^3 \times 3^3$.

Additional Puzzles

Puzzle: Consider a cell grid whose dimensions are $n \times n$. Using a Wunderlich cube (or a mystery cube) is it possible to construct an Eulerian cube-path whose stamping pattern can be connected to form a curve? Figure 14 shows an Eulerian cube-path over a 2×2 cell grid.

Figure 16: Wunderlich imprint.

Puzzle: Consider a rectangular cell grid whose dimensions are $n \times m$. For what values of n and m do complete cube-paths exist?

Puzzle: Orient the top face of a Wunderlich cube by placing a mark on one of the 4 corner areas of on the top face. Possible corners where marks can be placed are lower-left, lower-right, upper-left, and upper-right. This orientation mark can be created using a small dot-shaped or star-shaped sticker (or even a piece of tape). We say a cube has a mark on its lower-left corner if (1) looking down on the top face of the cube we see a mark on its lower-left corner, or (2) looking down from the top (through the cube) the bottom face of the cube has a mark on its lower-left corner.

Let $(cell_1, cell_2)$ denote a pair of cells belonging to a 3×3 cell grid, and let $(corner_1, corner_2)$ denote a pair of corners on a cube face. Does a cube-path exist beginning at $cell_1$ and ending at $cell_2$ such that when the cube is at $cell_1$ it has a mark on $corner_1$ and when the cube is at $cell_2$ it has a mark on $corner_2$?

Consider a 3×3 cell grid whose cells range over the coordinates $(0,0), \ldots, (2,2)$.

Example 1 *Question: Is it possible for a cube-path to begin at $(0,0)$ with the cube having a mark on its lower-left corner and end at $(1,0)$ with the cube having mark on its lower-right corner?*

Answer: Yes. One solution is $(0,0) \rightarrow (0,1) \rightarrow (1,1) \rightarrow (1,0)$.

Example 2 *Question: Is it possible for a cube-path to begin at $(0,0)$ with the cube having a mark on its lower-left corner and end at $(0,0)$ with the cube having the mark on its lower-right corner?*

Answer: Yes. One solution is $(0,0) \rightarrow (0,1) \rightarrow (0,2) \rightarrow (1,2) \rightarrow (1,1) \rightarrow (1,0) \rightarrow (0,0)$.

Conclusion

This paper argued that visual thinking and spatial abilities play an important role across a large number of disciplines including math and the sciences. A particularly important observation is that an individual can improve their visual thinking and spatial abilities by spending more time engaged in thought processes in which visual thinking and spatial reasoning play dominant roles. Combining art, math, and coding in a primary (and secondary school) curricula could be an effective way of teaching a variety of thought processes simultaneously. An important by-product of such an approach is that coding would then enter the educational mainstream in a manner that is minimally disruptive.

A visual thinking and spatial reasoning problem was then described involving the construction of an artifact we call the Wunderlich cube. Activities relating to the Wunderlich cube include (1) LEGO® constructions, (2) digital constructions, and (3) digitally fabricated constructions.

Acknowledgements

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